Method of Undetermined Coefficients (continued)

Summary of yesterday: For

$$ay'' + by' + cy = g(t)$$

if $g(t) = e^{\alpha t}\cos\beta t\,(a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0)$

$\qquad + e^{\alpha t}\sin\beta t\,(b_n t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0)$

where all a_n's and b_n's are known real numbers

then the first try template is

$$Y(t) = e^{\alpha t}\cos\beta t\,(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0)$$
$$\qquad + e^{\alpha t}\sin\beta t\,(B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0)$$

The complex number $\alpha + i\beta$ is called the exponent coefficient of $g(t)$ (or $Y(t)$).

Remark: The first try template depend only on $g(t)$. irrelevant of the left hand side.

* When the first try fails. multiply your template with a t, then try again.

Example: $y'' - y' - 2y = e^{2t}$

Char. eqn.: $r^2 - r - 2 = 0 \Rightarrow r_1 = 2,\ r_2 = -1$

Comp. sol'n : $y_c = C_1 e^{2t} + C_2 e^{-t}$

Try $Y = A e^{2t}$.

$Y'' - Y' - 2Y = 4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 0$

cannot be set equal to the RHS. First try fails.

Try $Y = Ate^{2t}$, $Y' = A(e^{2t} + t \cdot 2e^{2t}) = A(2t+1)e^{2t}$

$Y'' = A \cdot 2e^{2t} + A(2t+1) \cdot 2e^{2t} = A(4t+4)e^{2t}$

$Y'' - Y' - 2Y = Ae^{2t}(4t + 4 - (2t+1) - 2 \cdot t)$

$= A \cdot e^{2t} \cdot 3 = 3Ae^{2t}$

Set it equal to $e^{2t} \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$

$Y = \frac{1}{3} te^{2t}$

General solution : $y = C_1 e^{2t} + C_2 e^{-t} + \frac{1}{3} te^{2t}$.

<u>Example</u>: $y'' - 2y' + y = 2te^{t}$.

Char. eqn. : $r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$.

Comp. sol'n: $y_c = C_1 e^{t} + C_2 te^{t}$

First try: $Y = (At + B)e^{t}$. $Y' = Ae^{t} + (At+B)e^{t}$

$Y'' = Ae^{t} + Ae^{t} + (At+B)e^{t} = Ate^{t} + (2A+B)e^{t}$

$Y'' - 2Y' + Y = Ate^{t} + (2A+B)e^{t} - 2(Ate^{t} + (A+B)e^{t}) + (At+B)e^{t}$

$= (2A + B - 2A - 2B + B)e^{t} = 0$.

First try fail. Try $Y = (At+B)te^{t} = (At^2 + Bt)e^{t}$

$$Y' = (2At+B)e^t + (At^2+Bt)e^t = (At^2+(B+2A)t+B)e^t$$

$$Y'' = (2At+B+2A)e^t + (At^2+(B+2A)t+B)e^t$$

$$= (At^2+(B+4A)t+2B+2A)\,e^t$$

$$Y''-2Y'+Y = (\cancel{At^2}+\cancel{(B+4A)t}+2B+2A)\,e^t - \cancel{2}(\cancel{At^2}+\cancel{(2B+4A)t}+2B)e^t$$

$$+ (\cancel{At^2}+Bt)e^t$$

$= 2Ae^t$ cannot be set equal to $2te^t$.

Never regard $2A = 2t \Rightarrow A = t$. This is wrong because A should be a constant number.

* If second try fails, multiply your template with another t then try again.

Try $Y = (At^2+Bt)e^t \cdot t = (At^3+Bt^2)e^t$.

$$Y' = (3At^2+2Bt)e^t + (At^3+Bt^2)e^t = (At^3+(3A+B)t^2+2Bt)e^t$$

$$Y'' = (3At^2+(6A+2B)t+2B)e^t + (At^3+(3A+B)t^2+2Bt)e^t$$

$$= (At^3+(6A+B)t^2+(6A+4B)t+2B)e^t$$

$$Y''-2Y'+Y = t^3e^t(A-\underline{2}A+A) + t^2e^t(6A+B-\underline{2}(3A+B)+B)$$

$$+te^t(6A+4B-\underline{2}(2B)+0) + e^t\cdot 2B$$

$$= 6A\,te^t + 2B\,e^t$$

Set it equal to $2te^t \Rightarrow A = \frac{1}{3}$, $B = 0 \Rightarrow Y = \frac{1}{3}t^3e^t$

Gen. sol'n: $y = C_1e^t + C_2te^t + \frac{1}{3}t^3e^t$

Remark: For second order ODE, it won't fail more than twice.
The reason is seen later.

How to find the final template without trying.

Important fact: If the exponential coefficient of $g(t)$ appears in the list of characteristic roots for m times, then the first m tries fail.

In this case, the template should be set as $t^m \cdot$ (first try temp.)

Example: $y'' - y' - 2y = e^{2t}$

Char. roots $= 2, -1$.

exp. coeff of $g(t) = 2$, appearing as a single root

$\Rightarrow$ first try fails, the second try would succeed.

Should set $Y = t(Ae^{2t})$

It's precisely the $(m+1)$-th try that would succeed.

Example: $y'' - 2y' + y = 2te^t$

Char. roots $= 1, 1$.

exp. coeff. $= 1$ appears twice in the list of char. roots.

$\Rightarrow$ first & second try fail. third try succeeds.

$Y = t^2(At+B)e^t$.

Example: $y''+4y=t\sin 2t$ $\sin 2t = \mathrm{Im}.\ e^{2it}$

Char. eqn. $r^2+4=0 \Rightarrow r=2i,-2i$

Exp. coeff. $=2i$. appears once $\Rightarrow$ first try fail.

Set $Y=t\cdot[(At+B)\sin 2t+(Ct+D)\cos 2t]$

$=(At^2+Bt)\sin 2t+(Ct^2+Dt)\cos 2t.$

Example: $y''-2y'+2y=e^t\cos t.$ $e^t\cos t = e^t \,\mathrm{Re}\, e^{it} = \mathrm{Re}(e^{(1+i)t})$

Char. eqn: $r^2-2r+2=0 \Rightarrow r=\frac{2\pm\sqrt{2^2-8}}{2}=1\pm i$, $r=1+i, 1-i$.

Exp. coeff. $=1+i$. appears once $\Rightarrow$ first try fails

Set $Y=t(Ae^t\cos t+Be^t\sin t)$

$Y' = ((A+B)t+A)e^t\cos t+((B-A)t+B)e^t\sin t$

Do = by maple $Y'' = (2Bt+2A+2B)e^t\cos t+(-2At-2A+2B)e^t\sin t.$

$Y''-2Y'+2Y = 2Be^t\cos t-2Ae^t\sin t$

Set it equal to $e^t\cos t \Rightarrow 2B=1,\ 2A=0$

$\Rightarrow A=0,\ B=\frac{1}{2} \Rightarrow Y=\frac{1}{2}te^t\sin t$

$y=C_1e^t\cos t+C_2e^t\sin t+\frac{1}{2}te^t\sin t.$

When $g(t)$ is a sum of function of distinct exponent coefficients, you should deal with each summand independently, then add the sol'ns together. This works because of the following

Principle of Superposition (nonhomog. version):

If Y_1 is a solution to $y''+p(t)y'+q(t)y=g_1(t)$, and

Y_2 is a solution to $y''+p(t)y'+q(t)y=g_2(t)$,

then Y_1+Y_2 would be a solution to

$$y''+p(t)y'+q(t)y=g_1(t)+g_2(t).$$

Proof: The first sentence says $Y_1''+pY_1'+qY_1=g_1$

The second sentence says $Y_2''+pY_2'+qY_2=g_2$

Adding the equations: $Y_1''+Y_2''+pY_1'+pY_2'+qY_1+qY_2=g_1+g_2$

$$\Rightarrow (Y_1+Y_2)''+p(Y_1+Y_2)'+q(Y_1+Y_2)=g_1+g_2$$

$$\Rightarrow Y_1+Y_2 \text{ is a sol'n to } y''+py'+qy=g_1+g_2.$$

Example: $y''+y=t+t\sin t$.

Exp. coeff. of $t=0$

Exp. coeff. of $t\sin t=i$

Char. eqn: $r^2+1=0 \Rightarrow r=i,-i$

Set Y_1 to be a sol'n to $y''+y=t$

$Y_1=At+B$. $Y_1''=0$, $Y_1''+Y_1=At+B=t \Rightarrow A=1, B=0$

$\Rightarrow Y_1=t$.

Set Y_2 to be a sol'n to $y''+y=t\sin t$.

$$Y_2 = t\cdot\big((At+B)\sin t+(Ct+D)\cos t\big)$$

$$=(At^2+Bt)\sin t+(Ct^2+Dt)\cos t = Bt\sin t+Dt\cos t+\text{junk}$$

① Product rule of higher derivatives: $(fg)''=f''g+2f'g'+fg''$.

② RHS does not involve $t^2 \Rightarrow$ all terms of t^2 are junk.

$$Y_2''=2A\sin t+2\cdot(2At+B)\cos t+Bt(-\sin t)+\text{junk}$$

$$+2C\cos t+2\cdot(2Ct+D)(-\sin t)+Dt(-\cos t)+\text{junk}.$$

$$=\big[(4C-B)t+2A-2D\big]\sin t+\big[(4A-D)t+2B+2C\big]\cos t$$

$$Y_2''+Y_2=\big[4Ct+2A-2D\big]\sin t+\big[4At+2B+2C\big]\cos t$$

$$=t\sin t$$

$$\Rightarrow 4C=1,\ 4A-4D=0,\ 4A=0,\ 2B+2C=0$$

$$\Rightarrow A=0\quad B=-\frac{1}{4},\quad C=\frac{1}{4},\quad D=0$$

$$\Rightarrow Y_2=-\frac{1}{4}t\sin t+\frac{1}{4}t^2\cos t.$$

Gen. sol'n $y=y_c+Y_1+Y_2=C_1\cos t+C_2\sin t+t-\frac{1}{4}t\sin t+\frac{1}{4}t^2\cos t.$

Attendance Quiz: ① HW 3b: ω_0 real number. Find the general sol'n to the ODE: $y'' + \omega_0^2 y = \cos(\omega_0 t)$.

② Determine the template for the following ODE

(a) $y'' - 4y' + 5y = te^{2t}$

(b) $y'' - 6y' + 10y = t^2 e^{3t} \sin t$

(c) $y'' - 6y' + 9y = t^3 e^{3t}$.

(d) $y'' + y' = t^3 + e^{-t}$.

LECTURE NOTES OF DIFFERENTIAL EQUATION

The fastest way to find Y:

For $ay''+by'+cy=g(t)$.

Let $p(r)$ be the characteristic polynomial: $p(r)=ar^2+br+c$

Denote the differential operator by D, i.e., $D=\frac{d}{dt}$.

Example: $Dy=\frac{dy}{dt}=y'$, $D^2y=\frac{d^2y}{dt^2}=y''$

Observe that the left-hand-side can be written as

$$ay''+by'+cy=a\cdot D^2y+b\cdot Dy+c\cdot y$$
$$=(aD^2+bD+c\cdot 1)y$$
$$=p(D)y.$$

Here $p(D)$ is simply the characteristic polynomial with r evaluated as D.

The method of undetermined coefficient requires us to compute $aY''+bY'+cY$, which is precisely $p(D)Y$.

The following exponential shift lemma will greatly the computation:

For any function $f(t)$ and any polynomial $p(r)$,

$$p(D)(e^{\alpha t}f(t))=e^{\alpha t}p(D+\alpha)f(t)$$

Proof: We first look at the simplest case $p_0(r) = r$.

So $p_0(D) = D$. In this case the claim is precisely the product rule

$$D(e^{\alpha t} f(t)) = (e^{\alpha t} f(t))' = \alpha e^{\alpha t} f(t) + e^{\alpha t} f'(t)$$

$$= e^{\alpha t}(\alpha f(t) + f'(t))$$

$$= e^{\alpha t}(\alpha f(t) + D f(t)) = e^{\alpha t}(D+\alpha) f(t)$$

The next case : $p_n(r) = r^n$ can be proved by induction:

Indeed, if the claim is true for $p_{n-1}(r) = r^{n-1}$, then for r^n

$$D^n(e^{\alpha t} f(t)) = D \cdot D^{n-1}(e^{\alpha t} f(t))$$

induction hypo. $= D\left(e^{\alpha t}\left[(D+\alpha)^{n-1} f(t)\right]\right)$

product rule $= D(e^{\alpha t}) \cdot \left[(D+\alpha)^{n-1} f(t)\right] + e^{\alpha t} \cdot D\left[(D+\alpha)^{n-1} f(t)\right]$

$$= \alpha e^{\alpha t} \cdot \left[(D+\alpha)^{n-1} f(t)\right] + e^{\alpha t} \cdot \left[D(D+\alpha)^{n-1} f(t)\right]$$

$$= e^{\alpha t}\left(\alpha (D+\alpha)^{n-1} f(t) + D(D+\alpha)^{n-1} f(t)\right)$$

$$= e^{\alpha t} \cdot (D+\alpha)^n f(t).$$

Now that the claim is true for $p_n(r) = r^n$ for any n.

for generic polynomial $p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$,

$$p(D)\left[e^{\alpha t} f(t)\right] = \left(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0\right)\left[e^{\alpha t} f(t)\right]$$

$$= a_n D^n\left(e^{\alpha t} f(t)\right) + a_{n-1} D^{n-1}\left(e^{\alpha t} f(t)\right) + \cdots + a_1 D\left[e^{\alpha t} f(t)\right] + a_0 e^{\alpha t} f(t)$$

$$= a_n e^{\alpha t}(D+\alpha)^n f(t) + a_{n-1} e^{\alpha t}(D+\alpha)^{n-1} f(t) + \cdots + a_1 e^{\alpha t} D f(t) + a_0 e^{\alpha t} f(t)$$

$$= e^{\alpha t}\left[a_n(D+\alpha)^n + a_{n-1}(D+\alpha)^{n-1} + \cdots + a_1(D+\alpha) + a_0\right] f(t)$$

$$= e^{\alpha t} p(D+\alpha) f(t)$$

$\boxtimes$

We should see how this can be used in examples:

Example: $y'' + 2y' + y = (t^3 + 2t^2 + 5t + 1)e^{-t}$

Char. roots $= -1, -1 \Rightarrow y_c = C_1 e^{-t} + C_2 t e^{-t}$

Exp. coeff. $= -1$. First & second tries fail

Set $Y = t^2(At^3 + Bt^2 + Ct + E)e^{-t}$

$$= e^{-t}(At^5 + Bt^4 + Ct^3 + Et^2)$$

$$Y'' + 2Y' + Y = (D^2 + 2D + 1) Y$$

$$= (D+1)^2\left[e^{-t}(At^5 + Bt^4 + Ct^3 + Et^2)\right]$$

$$= e^{-t}(D-1+1)^2(At^5+Bt^4+Ct^3+Et^4)$$

$$= e^{-t}D^2(At^5+Bt^4+Ct^3+Et^2)$$

$$= e^{-t}(5\cdot 4At^3+4\cdot 3Bt^2+3\cdot 2Ct+2\cdot 1E)$$

Set it equal to $e^{-t}(t^3+2t^2+5t+1)$ to get

$$20A=1,\quad 12B=2,\quad 6C=5,\quad 2E=1$$

$$\Rightarrow A=\frac{1}{20},\quad B=\frac{1}{6},\quad C=\frac{5}{6},\quad E=\frac{1}{2}$$

$\Rightarrow$ Gen. sol'n: $y=C_1e^{-t}+C_2te^{-t}+\left(\frac{1}{20}t^5+\frac{1}{6}t^4+\frac{5}{6}t^3+\frac{1}{2}t^2\right)$

For $g(t)$ that involves trigs., in order to make use of the method, you need to consider the corresponding exponential function.

<u>Example</u>: $y''+y=(t^3+2t^2+5t+1)\sin t$

Char. eqn. $r^2+1=0 \Rightarrow r=-i,\ i.$

Comp. sol'n: $y_c=C_1\cos t+C_2\sin t.$

We consider the complexified ODE

$$y''+y=t^3e^{it}.$$

and try to find a particular complex solution:

Exp. coeff. of RHS = i. appears once in the list.

So first try fails.

Set $\tilde{Y} = t(At^3 + Bt^2 + Ct + D)e^{it}$

$$= e^{it}(At^4 + Bt^3 + Ct^2 + Dt)$$

$$\tilde{Y}'' + \tilde{Y} = (D^2+1)\left(e^{it}(At^4 + Bt^3 + Ct^2 + Dt)\right)$$

$$= e^{it}\left((D+i)^2 + 1\right)(At^4 + Bt^3 + Ct^2 + Dt)$$

$$= e^{it}(D+2i)D\,(At^4 + Bt^3 + Ct^2 + Dt)$$

$$= e^{it}(D+2i)(4At^3 + 3Bt^2 + 2Ct + D)$$

$$= e^{it}(12At^2 + 6Bt + 2C$$

$$+ 8iAt^3 + 6iBt^2 + 4iCt + 2iD)$$

$$= e^{it}\big(8iAt^3 + (12A + 6iB)t^2$$

$$+ (6B + 4iC)t + 2C + 2iD\big)$$

Set it equal to $e^{it}(t^3 + 2t^2 + 5t + 1)$ to get

$$8iA = 1.\quad 12A + 6iB = 2,\quad 6B + 4iC = 5,\quad 2C + 2iD = 1$$

$$\Rightarrow A = -\frac{1}{8}i,\ B = \frac{1}{6i}(2 - 12A) = \frac{1}{3i} - \frac{2}{i}\left(-\frac{1}{8}i\right) = \frac{1}{4} - \frac{1}{3}i$$

$$C = \frac{1}{4i}(5 - 6B) = \frac{5}{4i} - \frac{3}{2i}\left(\frac{1}{4} - \frac{1}{3}i\right) = \frac{1}{2} - \frac{7}{8}i$$

$$D=\frac{1}{2i}(1-2C)=\frac{1}{2i}\left(1-2\left(\frac{1}{2}-\frac{7}{8}i\right)\right)=\frac{7}{16}$$

So the complex solution is

$$\tilde{Y}=\left(-\frac{1}{8}i\,t^4+\left(\frac{1}{4}-\frac{1}{3}i\right)t^3+\left(\frac{1}{2}-\frac{7}{8}i\right)t^2+\frac{7}{16}t\right)e^{it}$$

$$=\left(-\frac{1}{8}i\,t^4+\left(\frac{1}{4}-\frac{1}{3}i\right)t^3+\left(\frac{1}{2}-\frac{7}{8}i\right)t^2+\frac{7}{16}t\right)(\cos t+i\sin t)$$

Since $\sin t$ is the imaginary part of e^{it}, to recover a real particular solution, we only need to find $\mathrm{Im}\,\tilde{Y}$

$$\mathrm{Im}\,\tilde{Y}=\left(+\frac{1}{8}\sin t\right)t^4+\left(\frac{1}{4}\sin t-\frac{1}{3}\cos t\right)t^3$$

$$+\left(\frac{1}{2}\sin t-\frac{7}{8}\cos t\right)t^2+\frac{7}{16}\sin t\ \ t$$

$$(a+bi)(\cos t+i\sin t)=(a\cos t-b\sin t)+i(b\cos t+a\sin t)$$

real part comes from yellow

imaginary part comes from brown

Lecture Notes of Differential Equation

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